Note Title 9/12/2006

Lecture 3

1) Basic notions from Math

2) Basic notions from computational geom

3) Graph search techniques $\longrightarrow$ Reading App. C from Latombe

better $\longrightarrow$ App. H from Choset

I) Basic Notions from Math

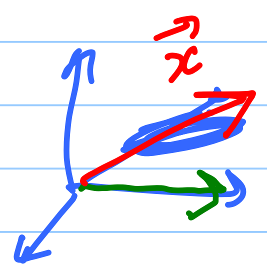
1) vector space : assumed known

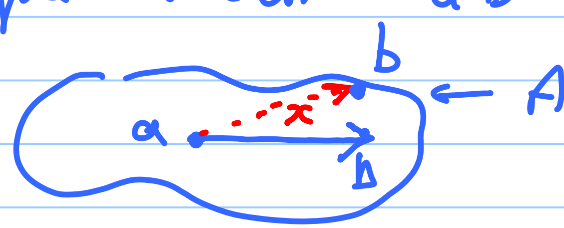
2) Affine space:

sets up a correspondence between "points" in a ~~space~~ set and . vectors in a vector space.

Let $\vec{E}$ be a vector space. The affine space associated with $\vec{E}$ is a set A:

1) every pair $(a, b) \in A \times A$ determines a unique vector $\vec{ab} \in \vec{E}$

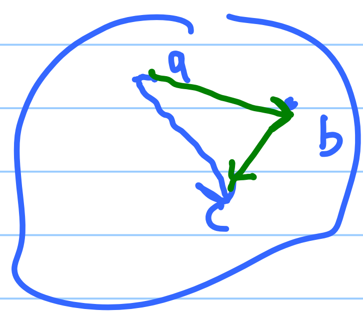




2) every pair $(a, \vec{x}) \in A \times \vec{E}$ determines a unique $b \in A$ such that $\vec{ab} = \vec{x}$

3) Law of parall.

$\forall\, a, b, c \in A$



$$\vec{ac} = \vec{ab} + \vec{bc}$$

elements of $\vec{A}$ are called points

one of the elements (arb. chosen) is called the origin, O. This allows a bijection

$$a \in A \Rightarrow \exists \vec{x} = \vec{Oa}$$

Inverse: for any $a \in A$, $c = -a$ is an element of A.

$$\vec{Oa} = \vec{x}$$

$$\vec{y} = -\vec{x} \qquad a' : \vec{Oa'} = -\vec{x}$$

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3) Metric or Distance over a set (assume a vector space E)

d is a metric or distance:

$$d : E \times E \longrightarrow \mathbb{R}$$

function ~~that~~ satisfies:

1) +veness : $\forall\, x, y \in E : d(x,y) \geq 0$

2) non-degenerate : $d(x,y) = 0 \iff x = y$

3) symmetric : $d(x,y) = d(y,x)$

4) triang. ineq: $d(x,y) + d(y,z) \geq d(x,z)$

" ~~small~~ "shortest"

E, d is a metric space.

Euclidean Dist. : $d(x,y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$

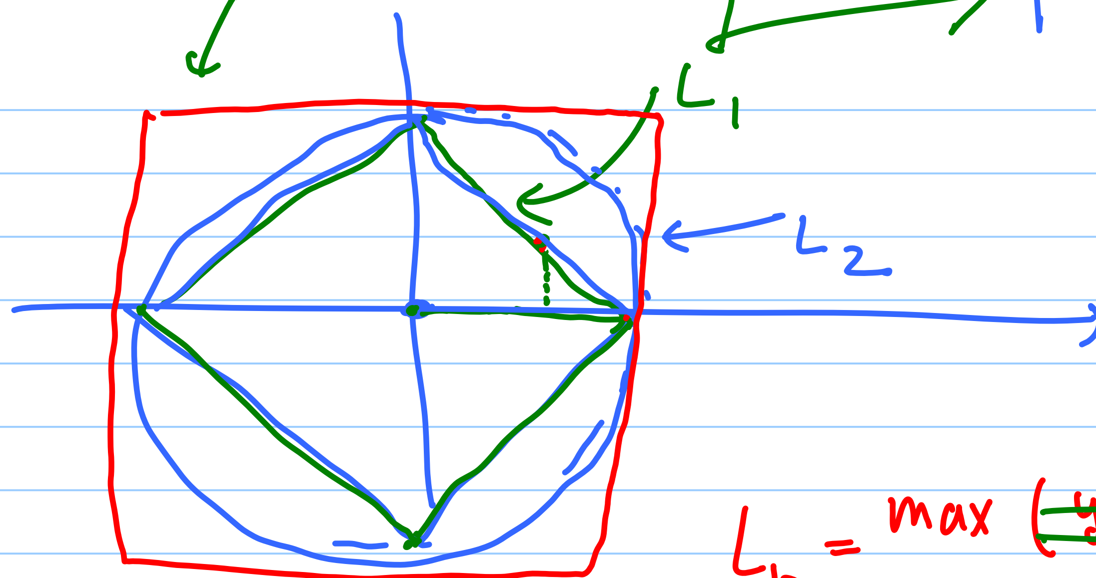
$x = (x_1, x_2)$
$y = (y_1, y_2)$

L_p metric:

$x = (x_1, \ldots x_m)$
$y = (y_1 \ldots y_m)$

$$d_p(x,y) = \sqrt[p]{\sum_{i=1}^{m} \left| x_i - y_i \right|^p}$$

$p = 1, 2, \infty$ it satisfies the above prop.



$|x_1 - y_1| + |x_2 - y_2|$

$x_1, x_2 = 0, 0$

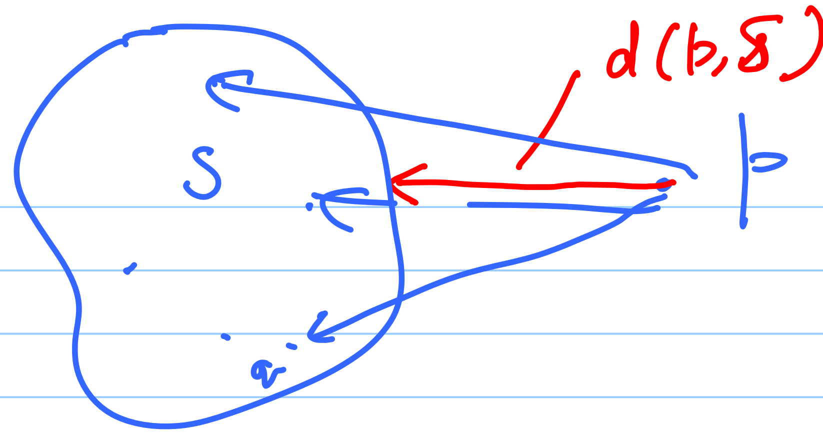
$|y_1| + |y_2|$

$L_p = \max(\cancel{y_1, y_2})$ max (x_1, x_2) or

for $p \to \infty$

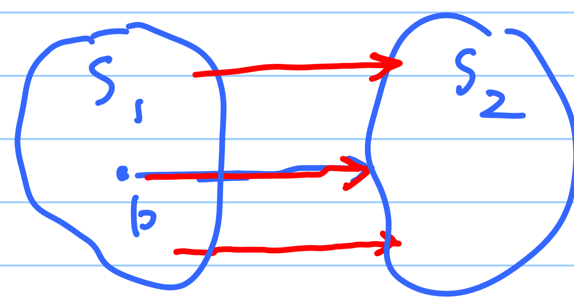
Distance between pt. & a set:

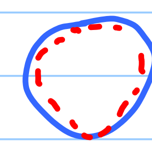
Compact set S



$$d(p, S) = \min_{q \in S} d(p, q)$$

<u>dist. bet. two sets S_1, S_2:</u>





Hausdorff distance:

$$\max_{p \in S_1} d(p, S_2)$$

is not symmetric: $d(S_1, S_2) \neq d(S_2, S_1)$

inner product: generalization of "angle"

$$a \cdot b = |a|\,|b| \cos\theta$$ (dot product)

$x \in E,\ y \in E$

$\langle x, y \rangle$ is a function: $E \times E \to R$

1) $\langle x, x \rangle \geq 0$

2) $\langle x, x \rangle = 0$ iff $x = \underline{0}$

3) $\langle x, y \rangle = \langle y, x \rangle$ symm

4) bilinear:

$$\langle \lambda_1 x_1 + \lambda_2 x_2, y \rangle = \lambda_1 \langle x_1, y \rangle + \lambda_2 \langle x_2, y \rangle$$

space all cont. func. $[0,1] \to R$

$$\langle f, g \rangle = \int_0^1 f(x) \cdot g(x)\, dx$$

$$\int_0^1 f^2(x)\, dx$$

✓ **Norm:** "length"

$x \in E \qquad \|x\| : E \to \mathbb{R}$

1) +ve : $\|x\| \geq 0$

2) non-deg: $\|x\| = 0$ iff $x = \underline{0}$

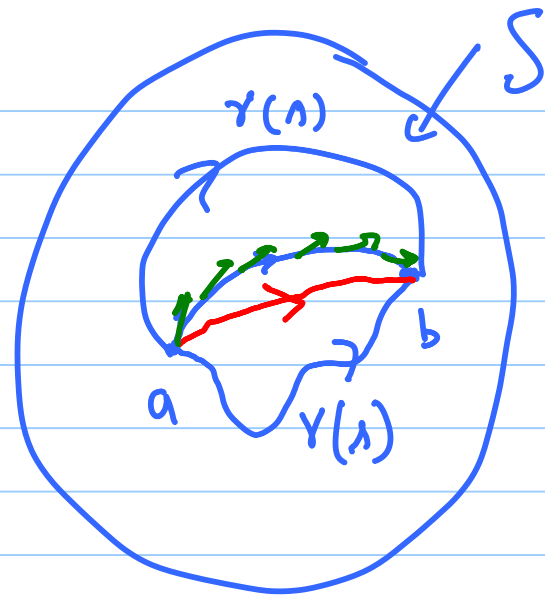
3) $\|\lambda x\| = |\lambda| \, \|x\|$

4) $\|x + y\| \leq \|x\| + \|y\|$

Inner product $\xrightarrow[\text{induces}]{}$ norm $\longrightarrow$ metric

$\langle x, x \rangle$

Surface
|| ger
manifold



Riemmanian Geometry

$\gamma(1) = b$
$\gamma(0) = a$

$\gamma : [0,1] \longrightarrow S$

$\gamma(s) \in S$

$\|\gamma\|$ $\quad L(\gamma) = \int_0^1 \left\| \frac{d\gamma(s)}{ds} \right\| ds$

$d(a,b) = \min_{\gamma} L(\gamma)$

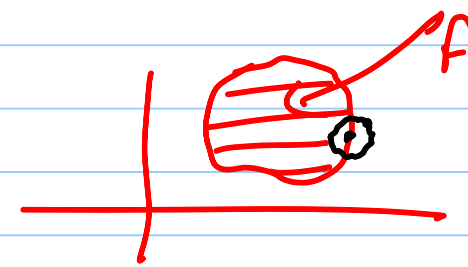
$\underline{\underline{R^n}}$

Elementary point-set topology : $x, y \in S$

assume notion of distance. $d(x,y)$ is given

$B_\varepsilon(p)$ = ε-ball : $\{ q \in S : d(p,q) < \varepsilon \}$

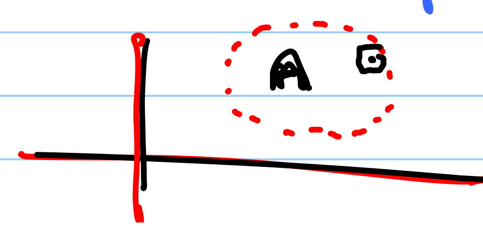
centered at p rad. ε ↔ ε-neighborhood



open set: $A \subset S$

1) A is open if every point in A has <u>an</u> ε-ball centered at it that belongs to A

$\varepsilon > 0$

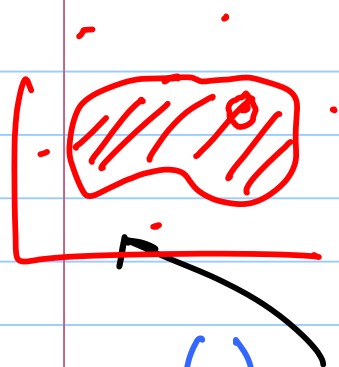


2) p : limit point of A : if every ϵ-ball centred at the point contains at least one pt. other than p $\in A$

Imp: limit pt. of A may not belong to A !!

3) Closure of a set A: $Cl(A) = A \cup$ limit pts. of A

4) closed set A : all limit pts of A $\in A$



5) Boundary pt of a set: a pt. is on bound. if every ε ball centered at pt. has at least one pt. $\in A$ and at least one pt. $\notin A$

6) interior pt. of A: $\exists$ an ε ball centered (at the pt) that $\subset A$

7) exterior pt. of A: $\exists$ an ε ball such that its int. with $A = \phi$

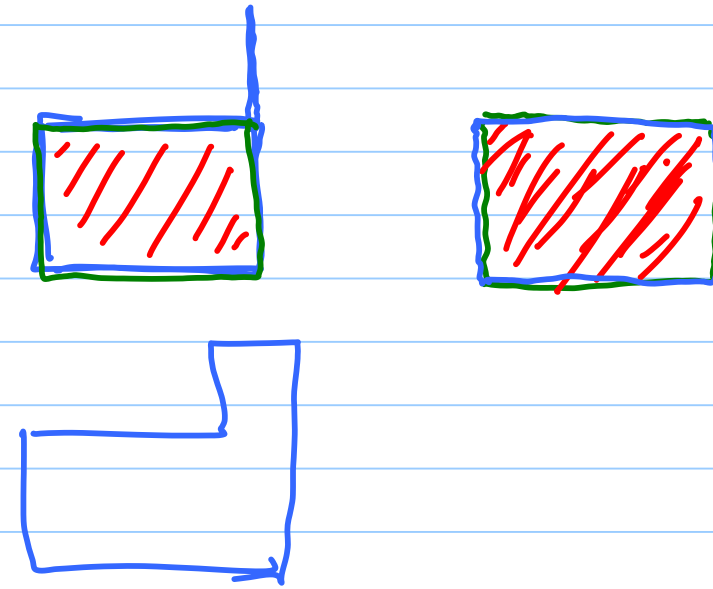
8) <u>Bounded set</u>: $A \subset R^n$

A is bounded if it is contained in some finite ball of R^n.

9) closed & bounded set : Compact Set

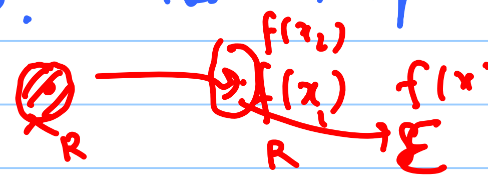
10) Regular set :

$$\boxed{A = Cl(int(A))}$$



11) Continuous map : its maps open sets $\longrightarrow$ open sets

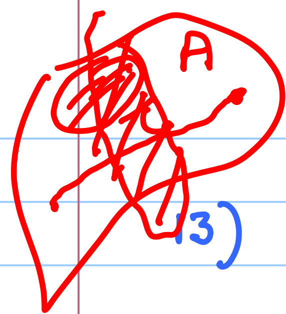
$f(x) : R \to R$



12) path : a Continuous map $\tau : [0,1] \longrightarrow A$

such that $\tau(0) = p \in A$

$\tau(1) = q \in A$



3) <u>path connected:</u>

A is path connected if given any two pts $\in A$ p, q

$\exists$ a path τ between them.

4) <u>homeomorphism</u> : X & Y are two open sets

$\phi : X \to Y$

mapping : 1) bijective $\to$ $1 \leftrightarrow 1$, $\to$ onto

$\underline{\underline{\phi}}$ 2) continuous

3) inverse is continuous

15) **diffeomorphism:** ① homeomorphism ② differentiable C^∞ → (smooth)

"smooth homomorphism"

C^k : k^{th} der. is continuous

$R^M \to R^n$

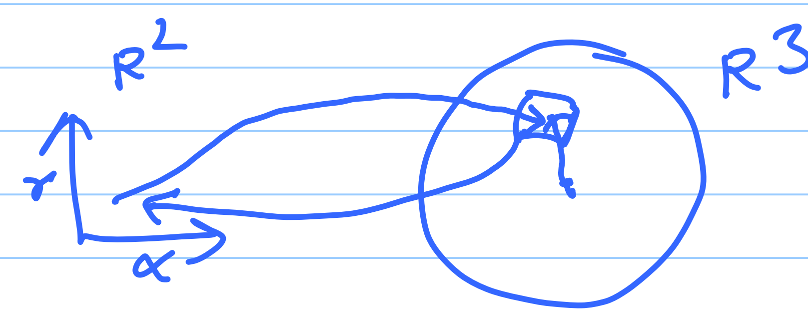
Manifolds: "generalization" of smooth surfaces

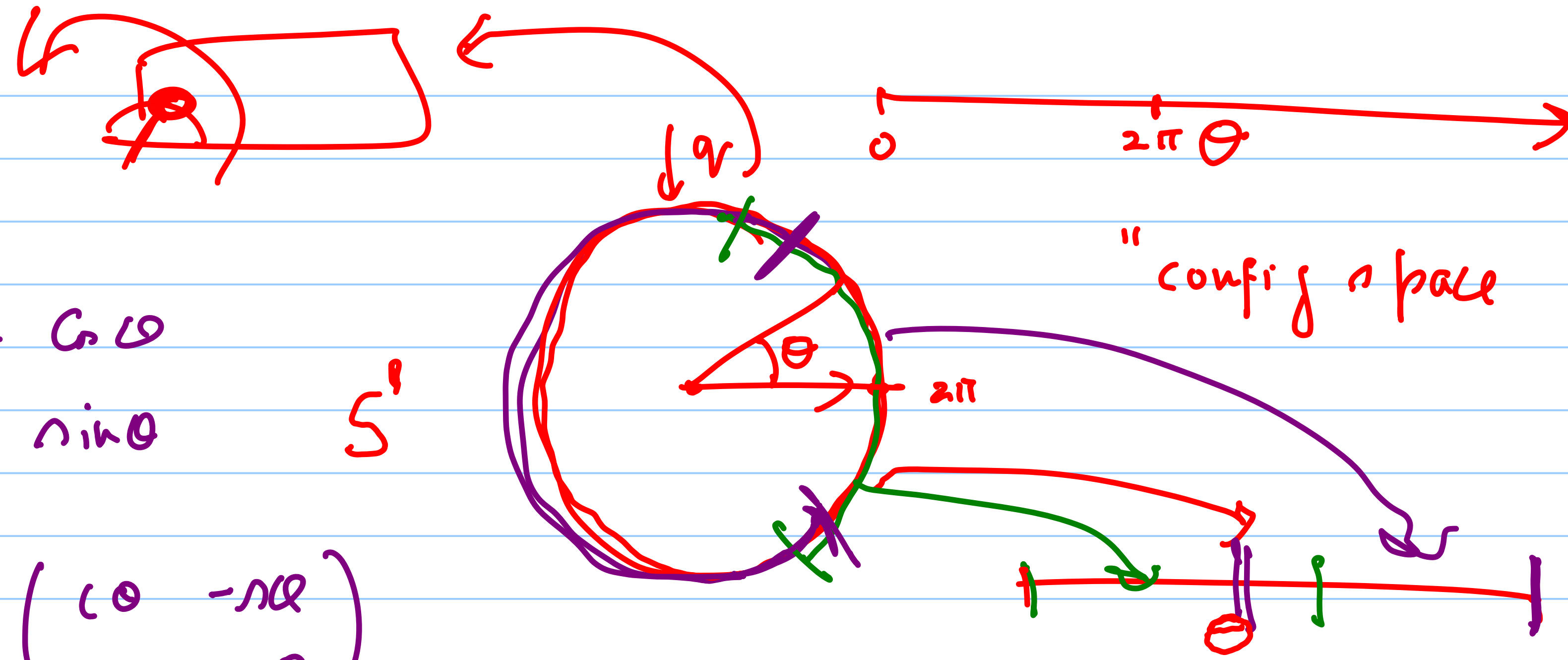
x: $\cos\alpha \sin$ - - -

y: - - -

Surface:

$$R^3 \underset{\phi^{-1}}{\overset{\phi_{\text{parame}}}{\rightleftarrows}} R^2 \quad \alpha, r$$





$x = \cos\theta$

$y = \sin\theta$

$$R_\theta \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$R \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$